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Universal Teichmüller space in geometry and physics

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Abstract

Lipman Bers' universal Teichmüller space, classically denoted by $T(1)$, plays a significant role in Teichmüller theory, because all the Teichmüller spaces $T(G)$ of Fuchsian groups G can be embedded into it as complex submanifolds. Recently, $T(1)$ has also become an object of intensive study in physics, because it is a promising geometric environment for a non-perturbative version of bosonic string theory. We provide a non-technical survey of what is currently known about the geometry of $T(1)$ and what is conjectured about its physical meaning. Our bibliography should be rather comprehensive, but we apologize for any unjustified omissions.

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Dedicated to the memory of Lipman Bers

1. Some classes of homeomorphisms

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. We shall denote the unit disc $\{z \in \widehat{\mathbb{C}} : |z| < 1\}$ by Δ, the unit sphere $\{z \in \widehat{\mathbb{C}} : |z| = 1\}$ by S^1, and the exterior of the unit disc $\{z \in \widehat{\mathbb{C}} : |z| > 1\} = \widehat{\mathbb{C}} \setminus (\Delta \cup S^1)$ by $\Delta^\star$.

A homeomorphism $w : D \to w(D)$ between domains in $\widehat{\mathbb{C}}$ is *quasiconformal* (qc) [3, 50] if and only if w has locally integrable generalized derivatives satisfying almost everywhere on D the *Beltrami equation*

$$w_{\bar{z}}(z) = \mu(z)\, w_z(z) \tag{1.1}$$

for some measurable complex function μ on D called the *Beltrami differential* with

$$\operatorname*{ess\,sup}_{z \in D} |\mu(z)| = \|\mu\|_\infty < 1. \tag{1.2}$$

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A solution of (1.1) is called *μ-conformal*; in the special case $\|\mu\|_\infty = 0$, w is *conformal*, i.e., biholomorphic.

Geometrically, quasiconformality means that f maps any infinitesimally small circle to an ellipse whose ratio of the major axis to the minor axis is uniformly bounded by some number $K < \infty$ called the *maximal dilatation.* Such an f is called *K-quasiconformal.* The relationship between the number K and the function μ, also called the *complex dilatation* of f, is as follows:

$$K = \sup_{z \in D} \frac{1 + |\mu(z)|}{1 - |\mu(z)|}. \tag{1.3}$$

Denote the space of Beltrami differentials $L^\infty(D)_1$; it is the open unit ball in the complex Banach space $L^\infty(D)$ of essentially bounded functions in D. The existence and uniqueness, up to three prescribed values, of solutions for the Eq. (1.1) with an arbitrary Beltrami differential is guaranteed by a fundamental theorem due to Gauss, Morrey, Bojarski [14], and Ahlfors and Bers [4]. In this survey, we omit most proofs, but many of the deeper aspects of the theory depend on careful analysis of the solutions of (1.1).

Let us think of the upper half-plane as a hemisphere of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{P}^1\mathbb{C}$. An increasing self-homeomorphism f of the real axis $\mathbb{R}$ is called *quasisymmetric* (qs) if it can be extended to a *quasiconformal* mapping of the upper half-plane $\mathbb{H}$ that fixes the point at infinity. Beurling and Ahlfors [13] showed that f is quasisymmetric, if for some constant K, $1 \leq K < \infty$,

$$\frac{1}{K} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq K \tag{1.4}$$

for all real x and positive t. More precisely, such an f is called *K-quasisymmetric* (K-qs).

We may always switch from $\mathbb{H}$ to Δ via the Cayley transform $z \mapsto (z-i)/(z+i)$, which maps $(0, 1, \infty)$ to $(-1, -i, 1)$, respectively. The explicit identification of $\mathbb{R}$ to S^1 is given by

$$x = -\cot \tfrac{1}{2}\theta, \qquad \text{or,} \qquad e^{i\theta} = \frac{x-i}{x+i}. \tag{1.5}$$

A continuous vector field $u(e^{i\theta})\, \partial/\partial\theta$ on S^1 becomes, on the real line, $F(x)\, \partial/\partial x$ with

$$F(x) = \tfrac{1}{2}(x^2+1)\, u\left(\frac{x-i}{x+i}\right). \tag{1.6}$$

Conversely,

$$u(e^{i\theta}) = \frac{2F(x)}{x^2+1} = 2\sin^2 \tfrac{1}{2}\theta\, F(-\cot \tfrac{1}{2}\theta). \tag{1.7}$$

In particular, if u vanishes at $(-1, -i, 1)$, we see that

$$F(0) = F(1) = 0 \quad \text{and} \quad \frac{F(x)}{x^2+1} \to 0 \text{ as } x \to \infty\,. \tag{1.8}$$

In analogy with the half-space model, an orientation-preserving self-homeomorphism f of the unit circle S^1 is called *quasisymmetric* (qs) if it can be extended to a quasiconformal (qc) mapping of the unit disc Δ. In the disc model, the K-qs condition is most conveniently given in terms of the cross ratio

$$(z_1, z_2, z_3, z_4) = \frac{z_4 - z_1}{z_4 - z_2} \frac{z_3 - z_2}{z_3 - z_1}. \tag{1.9}$$

We need to require that

$$\frac{1}{2K} \leq (f(z_1), f(z_2), f(z_3), f(z_4)) \leq 1 - \frac{1}{2K} \tag{1.10}$$

whenever $(z_1, z_2, z_3, z_4) = 1/2$. The explicit constant K may not be the same in (1.3), (1.4) and (1.10), but this is irrelevant for our present purposes. Again, a homeomorphism f is qs if it is K-qs for some K. In the sequel, we can equivalently deal with quasisymmetric maps on S^1 or $\mathbb{R}$, whichever seems to be more convenient, and we sometimes use the symbol X to designate either of these spaces. We shall then denote the group of qs maps of the space X by $\mathrm{QS}(X)$.

In the disc model, the Möbius group $\mathrm{M\ddot{o}b}(S^1)$ consists of the boundary transformations induced by the conformal automorphisms of Δ, the Möbius transformations $A : \Delta \to \Delta$,

$$Az = \lambda \frac{z - a}{1 - \bar{a}z} \tag{1.11}$$

with $z \in \Delta$, $|\lambda| = 1$ and $|a| < 1$. The Möbius group is a three-dimensional subgroup of $\mathrm{QS}(S^1)$ isomorphic to $\mathrm{PSU}(1,1;\mathbb{C})$.

Moreover, $\mathrm{M\ddot{o}b}(S^1)$ acts on the left on both $\mathrm{QS}(S^1)$ and $\mathrm{QC}(\Delta)$, the space of quasiconformal self-mappings of Δ. To see this, just notice that the cross ratio (1.9) is Möbius-invariant.

In the half-space model, the Möbius group is $\mathrm{PSL}(2;\mathbb{R})$, the group of maps of the form

$$f(z) = \frac{az + b}{cz + d} \tag{1.12}$$

with real coefficients a, b, c, d satisfying $ad - bc = 1$.

Möbius transformations can also be described as the solutions of the differential equation

$$Sf = 0, \tag{1.13}$$

where S is the *Schwarzian derivative*,

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{1.14}$$

Direct computation gives the transformation rule

$$S(f \circ g) = (Sf \circ g)(g')^2 + Sg, \tag{1.15}$$

so that precomposition by a Möbius transformation f leaves the Schwarzian derivative of a smooth function g invariant. The Schwarzian derivative will later show up in various contexts.

The extension of a qs map to a qc map is by no means unique. The extension operator constructed by Beurling and Ahlfors [14], who worked in the half-space model, has the drawback that it is not *conformally natural*. Working in the disc model, Tukia [80] and later also Douady and Earle [27] defined a conformally natural extension operator $E : \mathrm{QS}(S^1) \to \mathrm{QC}(\Delta)$ which satisfies the required naturality condition $E(A \circ f) = A \circ E(f)$ for any $A \in \mathrm{Möb}(S^1)$ and $f \in \mathrm{QS}(S^1)$. A simpler construction with refined estimates for the maximal dilatation has been given by Partyka [63]. A conformally natural extension operator is not unique either: another one $\tilde{E}$ is readily obtained by putting $\tilde{E}f = (E(f^{-1}))^{-1}$.

In particular, if f is a diffeomorphism on S^1, then it surely can be continued as a diffeomorphism, a fortiori, as a qc map, to the closed unit disc. Hence, smooth implies qs. We shall denote the group of C^∞ orientation-preserving diffeomorphisms of X by $\mathrm{Diff}(X)$.

The following chain of subgroup inclusions summarizes the most important classes of homeomorphisms:

$$\mathrm{Möb}(X) < \mathrm{Diff}(X) < \mathrm{QS}(X) < \mathrm{Homeo}(X). \tag{1.16}$$

Various other interesting spaces of homeomorphisms (real-analytic, Hölder, symmetric [34],...) could be designated. For our present purposes, let us introduce, following Zygmund [87], the Λ^* class

$$\begin{aligned}\Lambda^*(\mathbb{R}) = \{ F : \mathbb{R} \to \mathbb{R} \mid\ & F \text{ is continuous, satisfying normalizations (1.8); and,} \\ & |F(x+t) + F(x-t) - 2F(x)| \le C|t| \text{ for some constant } C, \\ & \text{for all } x \text{ and } t \text{ real.}\}\end{aligned}$$

Then $\Lambda^*(\mathbb{R})$ is a non-separable Banach space under the *Zygmund norm*, which equals, by definition, the best constant C for F. Namely,

$$||F|| = \sup_{x,t} \left| \frac{F(x+t) + F(x-t) - 2F(x)}{t} \right| \tag{1.17}$$

The interest of the *Zygmund class* $\Lambda^*(\mathbb{R})$ lies in the fact that, according to Reimann [72], the Zygmund class comprises precisely the vector fields for quasisymmetric flows on $\mathbb{R}$.

2. Geometric quantization of bosonic string theory

We shall now discuss some physics as a motivation for further mathematical developments. *Bosonic string theory* [26, 39, 60] is a proposal of unified field theory where the elementary particles called bosons are supposed to appear as one-dimensional extended objects on the Planck scale; hence, topologically they look like either $\mathbb{R}$ (open string) or S^1 (closed string). We shall work with closed strings. The string hypothesis introduces a new symmetry group into physics, the group Homeo(S^1), as this is the internal symmetry group of a closed string. *Non-perturbative bosonic string theory* would be based, ideally at least, on the group Homeo(S^1). We would like to geometrize this group, but as it seems to be intractable, in practice, we need to content ourselves with some subgroup.

There is a standard procedure in physics called *geometric quantization* [84] to pass from a classical system to a quantum system. In the classical system, the *observables* are functions f in the *phase space* which is a smooth manifold M^{2n} endowed with a symplectic form ω; in the corresponding quantum system the observables need to be converted into operators T_f acting in some Hilbert space in such a way that Poisson brackets of functions are converted into Lie brackets of operators

$$T_{\{f_1,f_2\}} = [T_{f_1}, T_{f_2}]. \tag{2.1}$$

The standard way to achieve this is to produce a Hermitian line bundle $\mathcal{L}$ over M with a Hermitian connection ∇ whose curvature equals ω. $\mathcal{L}$ exists if and only if ω represents an integral cohomology class. Then the sought-for operators will be given by

$$T_f = -i\nabla_{X_f} + f \tag{2.2}$$

where X_f is the Hamiltonian vector field corresponding to the observable f by the formula $X_f = -\omega^{-1}(df,\cdot)$. The operators T_f act in the Hilbert space of square-integrable sections of $\mathcal{L}$ with respect to the canonical volume form $\omega^n/n!$ of (M,ω). In fact, up to this point, we have only achieved *prequantization* while the difficult *Dirac problem* concerning the irreducibility of the representation $f \mapsto T_f$ remains to be settled. This final step in the geometric quantization programme can often be achieved by introducing a Kähler structure on the phase space and restricting to the holomorphic square-integrable sections.

Geometric quantization of string theory involves many unsolved problems which we shall discuss in due course later on. In any case, to get started we could try to produce a symplectic structure on Homeo(S^1) or, more modestly, on Diff(S^1). The Lie algebra of the infinite-dimensional Lie group Diff(S^1) is the algebra Vect(S^1) of smooth vector fields on the circle. These have Fourier modes labeled by the ring of integers $\mathbb{Z}$, so that Vect(S^1) formally behaves like an *odd*-dimensional space; hence, certainly Homeo(S^1) or Diff(S^1) as such cannot carry a symplectic structure. Heuristically, an odd number of degrees of freedom need to be removed before we can expect a symplectic phase space.

The simplest idea is to remove the Fourier zero mode by quotienting away the group of rotations Rot(S^1), or the circle itself. The resulting moduli space is denoted by

$$N = \mathrm{Diff}(S^1)/\mathrm{Rot}(S^1). \tag{2.3}$$

Bowick and Rajeev [17, 19-22] discovered that the space N carries, indeed, the structure of an infinite-dimensional Kähler manifold. We shall explicit this Kähler structure later on, but implicitly, this phenomenon may be understood as an infinite-dimensional analogue of the finite-dimensional standard argument of Kirillov, Kostant, and Souriau [46], which produces a symplectic structure in the coadjoint orbit space of a Lie group acting on the dual of its Lie algebra.

Moreover, there exists another obvious "even-dimensional" quotient space, namely

$$M = \mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1). \tag{2.4}$$

Then N is a holomorphic disc bundle over M. The Kirillov–Kostant–Souriau argument applies to M as well, and, indeed, Bakas [9] and Witten [83] have proved that the dual of the Lie algebra of $\mathrm{Diff}(X)$ admits no other coadjoint actions by non-trivial subgroups of $\mathrm{Diff}(X)$. Hence, in principle, we can choose either N or M as the underlying phase space of our geometric quantization scheme. We shall see that M is far more interesting.

We shall not review the method of coadjoint orbits, but let us mention that the dual of $\mathrm{Vect}(S^1)$ can be identified with the space of *Hill operators* H_u acting on smooth functions on the circle,

$$H_u = (d/dz)^2 + u(z), \tag{2.5}$$

where u is any smooth function on the circle [9]. Then, under arbitrary smooth reparametrizations $z \to f(z)$, the Hill operators transform as follows:

$$H_u \to \frac{1}{f'^2}\left(\frac{d}{dz}\right)^2 - \frac{f''}{f'^3}\frac{d}{dz} + u(f(z)). \tag{2.6}$$

Explicit calculation shows that (2.6) is equivalent to

$$H_u \to M(f'^{-3/2})\, H_{\tilde{u}}\, M(f'^{-1/2}), \tag{2.7}$$

where $M(\cdot)$ is the multiplication operator and $H_{\tilde{u}} = (d/dz)^2 + \tilde{u}(z)$ is given by

$$H_{\tilde{u}}(z) = (d/dz)^2 + f'^2 u(f(z)) + \tfrac{1}{2}\, Sf(z), \tag{2.8}$$

where S is the Schwarzian derivative (Segal [74]).

3. Universal Teichmüller space $T(1)$

Our fundamental sequence of inclusions (1.16) can be quotiented into

$$M = \mathrm{Diff}(X)/\mathrm{M\ddot{o}b}(X) \subset QS(X)/\mathrm{M\ddot{o}b}(X) \subset \mathrm{Homeo}(X)/\mathrm{M\ddot{o}b}(X). \tag{3.1}$$

Here we recognize a classical object; namely,

$$T(1) := QS(X)/\mathrm{M\ddot{o}b}(X) \tag{3.2}$$

is the *universal Teichmüller space* that Bers introduced in [10, 11]. Equivalently, we may think of $T(1)$ as the space of quasisymmetric homeomorphisms of the circle, say, which have three prescribed values. We stipulate the three points ± 1 and $-i$ to be fixed. The infinite-dimensional space $T(1)$ is *universal* in the sense that it contains as subspaces all the other Teichmüller spaces whose definition we briefly recall next [13, 43, 49, 54, 78].

Let G be a *Fuchsian group*, i.e., a discrete subgroup of $\text{Möb}(X)$. The *Teichmüller space* $T(G)$ is defined by

$$T(G) = \{[f] \in T(1) \mid f \circ \gamma \circ f^{-1} \in \text{Möb}(X) \text{ for all } \gamma \in G\}. \tag{3.3}$$

These spaces are partially ordered: $G < G'$ clearly implies $T(G') \subset T(G)$; in particular, all the Teichmüller spaces $T(G)$ are contained in the universal one. (The "1" in the notation $T(1)$ refers to the trivial group.) Moreover, the inclusion $T(G) \subset T(1)$ turns out to be a holomorphic embedding.

Another approach to Teichmüller theory is global analytic. The *Teichmüller space* $T(\Sigma)$ of a Riemann surface Σ is defined as the parametrization space of its complex structures up to isotopy. A complex structure may be given as a smooth section J of the endomorphism bundle of the tangent bundle of the surface Σ which is an anti-involution, i.e.,

$$J^2 = -\,\text{id}. \tag{3.4}$$

Denote the space of all complex structures J by $\mathcal{A}$. Of course, $\text{Diff}(\Sigma)$ acts on $\mathcal{A}$ via the pull-back operation. In the case of a compact orientable surface of genus > 1, the Teichmüller space $T(\Sigma)$ simply equals the moduli space $\mathcal{A}/\text{Diff}_0(\Sigma)$ where $\text{Diff}_0(\Sigma)$ is the identity component of $\text{Diff}(\Sigma)$. This simple global analytic definition is powerfully exploited in the treatise [78].

In the above set-up, we can easily define a Kähler structure on $T(\Sigma) = \mathcal{A}/\text{Diff}_0(\Sigma)$. By differentiating the relation (3.4), we see that the tangent vectors $\dot{J}$ of $T(\Sigma)$ at J anticommute with J under composition of endomorphisms. Hence, the formula

$$\omega(\dot{J}_1, \dot{J}_2) = \int_{\Sigma} \text{tr}\,(J \circ \dot{J}_1 \circ \dot{J}_2) \tag{3.5}$$

defines a 2-form on $\mathcal{A}$. The integration is with respect to the hyperbolic metric uniquely corresponding to J. This correspondence is natural with respect to the action of $\text{Diff}(\Sigma)$; a fortiori, with respect to the action of the subgroup $\text{Diff}_0(\Sigma)$, so it passes to the quotient. Moreover, it is straightforward to check that the resulting 2-form on the Teichmüller space $T(\Sigma)$ is non-degenerate and closed; hence, a Kähler form. This Kähler form (up to a constant multiple) is the classical Weil–Petersson Kähler 2-form.

If the surface Σ is uniformized by a Fuchsian group G, then the two definitions coincide:

$$\Sigma = X/G \quad \Rightarrow \quad T(\Sigma) = T(G). \tag{3.6}$$

This is a theorem of Tukia [80], which Douady and Earle [27] reproved using their conformally natural extension operator.

We shall briefly indicate how $T(\Sigma)$ and $T(G)$ are related to each other. For this purpose, we need to discuss two pertinent classes of solutions for the Beltrami Equation (1.1) corresponding to specially chosen Beltrami differentials $\mu \in L^\infty(\widehat{\mathbb{C}})_1$.

The real-analytic w_μ-theory: By applying the fundamental existence and uniqueness theorem to the Beltrami differential which is μ on Δ and is extended to Δ^* by reflection $[\tilde{\mu}(1/\bar{z}) = \overline{\mu(z)} z^2/\bar{z}^2$ for $z \in \Delta]$, one obtains the quasiconformal homeomorphism w_μ of $\mathbb{C}$ which is μ-conformal in Δ, fixes ± 1 and $-i$, and keeps Δ and Δ^* both invariant.

The complex-analytic w^μ-theory: By applying the existence and uniqueness theorem to the Beltrami differential which is μ on Δ and zero on Δ^*, one obtains the quasiconformal homeomorphism w^μ on $\mathbb{C}$, fixing $0, 1, \infty$, which is μ-conformal on Δ and conformal on Δ^*.

It is a fact that w_μ depends only real-analytically on μ, whereas w^μ depends complex-analytically on μ. The latter extension is so useful that it carries the name "Bers' trick".

Let us now try to make (3.6) look a little more plausible. First of all, a Beltrami differential μ is *G-equivariant*, if it is compatible with the action of G on Δ; more precisely, this leads to the requirement

$$\mu(\gamma z)\bar{\gamma}'(z)/\gamma'(z) = \mu(z), \tag{3.7}$$

which should hold almost everywhere on Δ for every $\gamma \in G$. Let us denote the space of G-compatible Beltrami differentials $L^\infty(G)$. An alternative description of $T(G)$ can now be given as

$$T(G) = L^\infty(G)/\sim, \tag{3.8}$$

where $\mu \sim \nu$ if and only if $w_\mu = w_\nu$ on $\partial\Delta = S^1$, which happens if and only if $w^\mu = w^\nu$ on $\Delta^* \cup S^1$.

Now, if μ is G-invariant, then w_μ conjugates G to another Fuchsian group

$$G_\mu = w_\mu G w_\mu^{-1}. \tag{3.9}$$

The equivalence class of μ in $T(G)$ represents the Riemann surface $X_\mu = \Delta/G_\mu$.

In the reverse direction, one can use w^μ to conjugate G to a *quasi-Fuchsian group*

$$G^\mu = w^\mu G(w^\mu)^{-1}, \tag{3.10}$$

so that G^μ acts discontinuously on the *quasidisc* $\Delta^\mu = w^\mu(\Delta)$ and its exterior $\Delta^{*\mu} = w^\mu(\Delta^*)$. Now, the Riemann surface X_μ is represented by Δ^μ/G^μ (whereas $\Delta^{*\mu}/G^\mu$ is the fixed Riemann surface Δ^*/G, since w^μ is conformal on Δ^*).

4. Models of $T(1)$

We can think about $T(1)$ in several ways. The following three classical models of $T(1)$ are the best-known:

(a) *the real-analytic model* consisting of all Möbius-normalized quasisymmetric homeomorphisms of the unit circle S^1;

(a′) *the geometric model* consisting of all Möbius-normalized *quasicircles*, i.e., all images of the standard circle under a global quasiconformal map that fixes the points ± 1 and $-i$;

(b) *the complex-analytic model* comprising all functions which fix $0, 1, \infty$, which are univalent on the exterior of the unit disc Δ^* and which allow quasiconformal extension to the whole Riemann sphere.

For yet other models, see [45, 47].

Specializing (3.8), we may also define the universal Teichmüller space as a quotient of Beltrami differentials:

$$T(1) = L^\infty(\Delta)_1/\sim\,, \tag{4.1}$$

where $\mu \sim \nu$ if and only if $w_\mu = w_\mu$ on $\partial\Delta = S^1$, or equivalently, if and only if w^μ and w^ν coincide on $\Delta^* \cup S^1$.

We let

$$\Phi : L^\infty(\Delta)_1 \longrightarrow T(1) \tag{4.2}$$

denote the quotient projection. $T(1)$ inherits its canonical structure as a complex Banach manifold from the complex structure of $L^\infty(\Delta)_1$; indeed, Φ becomes a holomorphic submersion.

The derivative of Φ at $\mu = 0$:

$$d_0\Phi : L^\infty(\Delta) \longrightarrow T_0T(1)\,, \tag{4.3}$$

is a complex-linear surjection whose kernel is the space N of "infinitesimally trivial Beltrami differentials"

$$N = \left\{ \mu \in L^\infty(\Delta) : \int_\Delta \mu\phi = 0 \quad \text{for all} \quad \phi \in A(\Delta) \right\}, \tag{4.4}$$

where $A(\Delta)$ is the Banach space of L^1 integrable holomorphic functions on the disc. Thus, the tangent space at the origin $0 = \Phi(0)$ of $T(1)$ is $L^\infty(\Delta)/N$.

It is now clear that to $\mu \in L^\infty(\Delta)_1$ we can associate the quasisymmetric homeomorphism

$$f_\mu = w_\mu|_{S^1} \tag{4.5}$$

as representing the Teichmüller point $[\mu]$ in the real-analytic model (a) of $T(1)$. Indeed $T(1)_{(a)}$ is the homogeneous space:

$$\begin{aligned} T(1)_{(a)} &= \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1) \\ &= \{\text{quasisymmetric homeomorphisms of } S^1 \text{ fixing } \pm 1 \text{ and } -i\}\,. \end{aligned}$$

In the geometric model (a′) of $T(1)$, we think of the points of $T(1)$ as the images of S^1 under w^μ.

There is a plethora of characterizations of the quasidiscs and their boundaries, the quasicircles [37]. Perhaps the most elegant is Ahlfors' condition [2], which identifies quasicircles among those Jordan curves of the complex plane which pass through ∞ (this can be achieved by a Möbius transformation): Such a Jordan curve C is a quasicircle if and only if there is a constant M such that for any three distinct points a, b, c on C with b between a and c

$$|b-a| \leq M|c-a|. \tag{4.6}$$

A generic quasidisc turns out to be a fractal object.

Alternatively, $[\mu]$ is represented by the univalent function

$$f^{\mu} = w^{\mu}|_{\Delta^{\star}} \tag{4.7}$$

on $\Delta^{\star}$, in the complex-analytic model (b) of $T(1)$. A more natural choice of the univalent function representing $[\mu]$ is to use a different normalization for the solution w^{μ} (since we have the freedom to post-compose by a Möbius transformation). In fact, let

$$W^{\mu} = M^{\mu} \circ w^{\mu}, \tag{4.8}$$

where M^{μ} is the unique Möbius transformation so that the univalent function (representing $[\mu]$):

$$F^{\mu} = W^{\mu}|_{\Delta^{\star}} \tag{4.9}$$

has the properties:

(i) F^{μ} has a simple pole of residue 1 at ∞ ;

(ii) $(F^{\mu}(z) - z) \to 0$ as $z \to \infty$.

Thus, the expansion of F^{μ} in $\Delta^{\star}$ is of the form:

$$F^{\mu}(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots . \tag{4.10}$$

Let us note that the original ($0, 1, \infty$ fixing) normalization gives an expansion of the form:

$$f^{\mu}(z) = z\left(a + \frac{\beta_1}{z} + \frac{\beta_2}{z^2} + \frac{\beta_3}{z^3} + \cdots\right). \tag{4.11}$$

and the Möbius transformation M^{μ} must be $M^{\mu}(w) = w/a - \beta_1/a$. Since $(a, \beta_1, \beta_2, ...)$ depend holomorphically on μ, we see that $(b_1, b_2, b_3, ...)$ also depend holomorphically on μ. Thus, our complex-analytic version of $T(1)$ is:

$$T(1)_{(b)} = \{\text{Univalent functions in } \Delta^{\star} \text{ with power series of the form (4.10),}$$
$$\text{allowing quasiconformal extension to the whole plane}\}.$$

In the general theory of univalent functions, the functions of the type (4.10) are known as the class Σ [28]. It is not difficult to compute that the area of the corresponding quasidisc is

$$A = \pi \left(1 - \sum_{n=1}^{\infty} n \, |b_n|^2 \right) . \tag{4.12}$$

Of course, this is non-negative so that we deduce the classical Area Theorem [28, 49, 54] about the coefficients b_n in the class Σ:

$$\sum_{n=1}^{\infty} n \, |b_n|^2 \leq 1. \tag{4.13}$$

We may think of the coefficients b_n as coordinates on $T(1)$. A refinement of the Area Theorem shows that $b_n = \mathcal{O}(n^{-c})$ with $c = 0.509...$ [23], but the coefficients in the class Σ still retain many mysteries. It is known that $|b_1| \leq 1$, $|b_2| \leq \frac{2}{3}$, and $|b_3| \leq \frac{1}{2} + e^{-6}$. These bounds are *sharp* but there is no "Bieberbach conjecture" about the general sharp upper bound for $|b_n|$. Nonetheless, we can think of $T(1)$ as a space of certain sequences $(b_1, b_2, b_3, \ldots)$.

5. The physicist's wish-list

In this chapter, we explain why the universal property of $T(1)$ makes it an attractive object of study from the point of view of *non-perturbative* bosonic string theory, whose precise geometric formulation, as we should stress, is unknown. First we shall review in simple non-technical terms the basic ideas of the prevailing *perturbative* bosonic string theory to the benefit of the reader who is not familiar with the physics literature.

The central issue in any quantum field theory is to evaluate the *partition function* Z, which gives the quantum-mechanical probability amplitudes of the system under study. Feynman introduced in 1948 a quantization scheme where Z is computed as a *path integral* over the space of paths representing the possible worldlines of elementary particles. The possible spacetime trajectories of a propagating pointlike elementary particle are one-dimensional paths, whereas a propagating bosonic *string*, or a one-dimensional extended object, sweeps out two-dimensional world-surfaces. A natural generalization of the Feynman path integral then is an integral over all possible world-sheets. As a first approximation, one can limit to deal with compact orientable surfaces which are topologically classified by the genus $\gamma = 0, 1, 2, \ldots$ The emergence of a handle in the propagation pattern corresponds to the breaking apart of two strings; correspondingly, the annihilation of two strings closes a handle. In reality, one should also take into account the degenerate situations where, e.g., a handle shrinks to a node.

While the topological classification of compact orientable surfaces is easily understood, their geometrical diversity is more intricate. The possible geometries are given as the infinite-dimensional cone $\mathcal{M}$ of all Riemannian metrics g of the underlying topological surface. This space, however, is physically redundant. The physically meaningful space in each genus γ is the parametrization space of conformal structures, or, the *Riemann moduli space* $\mathcal{M}_\gamma$. It is well-known that $\mathcal{M}_0$ is a point, $\mathcal{M}_1 = \mathbb{H}/\mathrm{PSL}(2;\mathbb{Z})$, while $\mathcal{M}_\gamma$ is an orbifold of dimension $6\gamma - 6$. The Riemann moduli space of a surface Σ admits as its covering space the *Teichmüller space* $T(\Sigma)$, the parametrization space of conformal structures up to isotopy. Conformal structures are the relevant *intrinsic* geometries of the surface Σ, so one should develop an integration scheme over the moduli. Some explicit results are known, e.g., the volumes of some Teichmüller spaces in the Weil–Petersson metric (Penner [66]).

We think of the string propagating in a fixed background space that, as a first approximation, can be taken to be the flat Minkowski space of some dimension D. We also need to take into account the *extrinsic* geometries of the string; in other words, the various ways in which a string may be embedded into the ambient spacetime. An extrinsic metric is induced on the surface Σ as a pull-back of the flat background metric via an embedding s. Thus we should also integrate over all embeddings $s : \Sigma \to \mathbb{R}^{D-1,1}$.

Consequently, we should be looking for the partition function in the form of a perturbative series

$$Z = \sum_{\gamma=0}^{\infty} \int_{\mathcal{M}_\gamma} e^{-S}. \tag{5.1}$$

Here $S = S(g, s)$ is the *Polyakov energy*, i.e., the Dirichlet energy of an arbitrary embedding of the propagating string into the background spacetime. The integration in (5.1) is with respect to the so-called *Polyakov measure* over the moduli space $\mathcal{M}_\gamma$ of each genus γ and also over the infinite-dimensional space of all embeddings s. Polyakov discovered in 1981 that this measure exhibits *conformal anomaly cancellation* in the critical spacetime dimension $D = 26$.

Perturbative bosonic string theory suffers from several drawbacks. First of all, the summation over the genus in (5.1) is well-known to be divergent [40, 66]. Secondly, the need to prescribe the topology and geometry of the background spacetime is philosophically unsatisfactory. The spacetime should rather arise as an excitation. So far, merely the critical dimension $D = 26$ arises as a constraint. Thirdly, the critical dimension is outlandish, be it lowered to the slightly more palatable $D = 10$ in *superstring theory* [26, 39] which incorporates fermions as well.

Perhaps these drawbacks indicate that we are just scratching the surface of some underlying intrinsic geometric principle that would imply more stringent conditions on the global properties of spacetime. The proper geometric environment of bosonic string theory should be some kind of "universal Riemann moduli space" which would comprise the moduli of surfaces with an arbitrary number of handles, cusps, boundary components, and nodes. One heuristic candidate for such an object has been put forward by Friedan and Shenker [32, 33], but it is not mathematically well-established.

The only classically known universal moduli space in mathematics literature is Bers' universal Teichmüller space $T(1)$, although no viable notion of universal Riemann moduli space corresponds to it. The potential physical interpretation of $T(1)$ as a superspace, in the sense of DeWitt and Wheeler, was discussed by Bers already in [12]. From the modern point of view, $T(1)$ is bound to be a highly relevant object, as it plays a role in both of the existing approaches to the quantization of bosonic strings:

(i) From the point of view of *perturbative* bosonic string theory, $T(1)$ contains as subspaces all the finite-dimensional Teichmüller spaces corresponding to various perturbative orders.

(ii) From the point of view of *non-perturbative* bosonic string theory, $T(1)$ is contained in the space $\mathrm{Homeo}(S^1)/\mathrm{Möb}(S^1)$ which, in principle, should be the ultimate arena of the geometric quantization of bosonic string theory.

Perhaps the perturbative series (5.1) ought to be replaced by a single integral over the universal Teichmüller space

$$Z = \int_{T(1)} e^{-S}. \tag{5.2}$$

Some preliminary speculations about how the measure in $T(1)$ should look like in terms of the coefficients b_n appear in [42], but no actual progress has been recorded.

The physicists' wish-list for mathematicians to achieve the geometric quantization of $T(1)$ includes (at least) the following items [67]:

(1) *Universal geometry:* $T(1)$ should be a Kähler manifold whose Kähler form ω pulls back to the Weil–Petersson form on each classical Teichmüller space $T(G)$.

(2) *Universal topology:* There should be an action by a "universal mapping class group" on $T(1)$ which pulls back to each $T(G)$.

(3) *Universal line bundle:* There should exist a Hermitian line bundle $\mathcal{L}$ over $T(1)$ with a connection whose curvature equals ω.

(4) *Universal measure:* $T(1)$ should carry a "Haar measure" with respect to which the classical locus, or the union of the images of the embeddings of all the $T(G)$, is dense and of measure zero in $T(1)$.

(5) *Universal action principle:* There should exist a scalar-valued function S ("universal Polyakov energy") whose gradient flow should have a superset of the classical locus as attracting fixed points. This function should coincide with a Kähler potential of the universal Weil–Petersson Kähler form.

We shall review the above-listed items emphasizing the established aspects of the theory. The current state of art seems to be that the item (1) is well-established for the quotient space $\mathrm{Diff}(S^1)/\mathrm{Möb}(S^1)$ while some evidence of its validity has been advanced in a suitable subspace of $T(1)$ (Nag and Sullivan [58]) and, from a different point of view, even in $\mathrm{Homeo}(S^1)/\mathrm{Möb}(S^1)$ (Penner [67]).

The solution to the item (2) has been claimed by Penner [67], but the discussion of his graph theoretic methods would bring us too far. Ratiu and Todorov [71] suggested that Quillen's determinant line bundle construction [70] applied to a family of Cauchy–Riemann operators parametrized by M could solve the item (3). However, one sees with difficulty how Quillen's construction could be extended to $T(1)$. Possibly, ordinary calculus ought to be replaced by "quantum calculus" in the sense of Alain Connes' non-commutative geometry [24].

The item (4) has been preliminarily discussed by Wiesbrock [81, 82] and by Nag and

Sullivan [58]. Some time ago, we suggested in [65] that the quasidisc area functional A on $T(1)$ might serve as a heuristic candidate for the universal action principle required in item (5), because the quadratic expression in (4.12) can be interpreted as the Dirichlet energy of the harmonic extension in Δ defined by the boundary values of the univalent function F^{μ} in (4.10). However, we have not been able to compare the functional A with the Polyakov energy of each genus.

We have not yet pointed out the existence of a natural distance function on $T(1)$. Denote by $K(h)$ the minimal dilatation of a quasiconformal self-mapping of Δ with the same boundary values as h. Then the *Teichmüller metric* on $T(1)$ is defined by

$$d_1(f,g) = \tfrac{1}{2}\log K(f\circ g^{-1}), \quad [f],[g] \in T(1). \tag{5.3}$$

Obviously, the value $d_1(f,g)$ does not change if we replace f by $A\circ f$ and g by $B\circ g$, where A and B are conformal mappings, and hence the Teichmüller metric is well-defined. Minimizing the dilatation only over G-compatible quasiconformal mappings, one analogously obtains a Teichmüller metric d_G for each $T(G)$. On the other hand, the metric space $(T(1), d_1)$ induces a metric space structure to each subspace $T(G)$. The Teichmüller metric d_1 on $T(1)$ is universal in the sense that it induces the same topology as d_G to each $T(G)$. Moreover, $d_1 \leq d_G$ and, according to Strebel [77], in general, $d_1 < d_G$. The Weil–Petersson metric is non-complete, while the Teichmüller metric is complete. In particular, the two metrics are not equivalent. Indeed, there is no Riemannian metric corresponding to the Teichmüller metric, so that it does not provide an answer to the item (1) in the physicists' wish-list above.

The introduction of Diff S^1 in string theory was originally motivated as a globalization of the work of Frenkel, Garland, and Zuckerman [31], who gave the conditions for the consistency of string theory in terms of a certain Lie algebra cohomology of vector fields of the circle. The *algebraic* approach to string theory is a vast topic which is beyond the scope of this survey. Let us mention, though, that the need to understand better also the algebraic relationship between Polyakov's perturbative approach and the non-perturbative geometric quantization approach has been emphatically expressed by Manin [51].

6. The tangent space of $T(1)$

In order to do differential geometry on $T(1)$, we first need to describe its tangent space in the various models.

6.1. Tangent space to the real-analytic model

Since $T(1)$ is a homogeneous space (according to the model (a)) for which the right translation by any fixed quasisymmetric homeomorphism acts as a biholomorphic automorphism, it is enough in all that follows to restrict attention to the tangent space at a single point of $T(1)$, the origin, or, the class of the identity homeomorphism.

Given any $\mu \in L^\infty(\Delta)$, the tangent vector $d_0\Phi(\mu)$ is represented by the real vector field $V[\mu] = \dot{w}[\mu]\partial/\partial z$ on the circle that produces the one-parameter flow $w_{t\mu}$ of quasisymmetric homeomorphisms:

$$w_{t\mu}(z) = z + t\dot{w}[\mu](z) + o(t)\,. \tag{6.1}$$

The vector field becomes in the θ-coordinate:

$$V[\mu] = \dot{w}[\mu](z)\frac{\partial}{\partial z} = u(e^{i\theta})\frac{\partial}{\partial\theta}, \tag{6.2}$$

where

$$u(e^{i\theta}) = \frac{\dot{w}[\mu](e^{i\theta})}{ie^{i\theta}}\,. \tag{6.3}$$

By our normalization, u vanishes at ± 1 and $-i$.

As mentioned before, the Zygmund class $\Lambda^*(\mathbb{R})$ comprises precisely the vector fields for quasisymmetric flows on $\mathbb{R}$. Hence, the tangent space to the real-analytic model of $T(1)$ becomes:

$$T_0(T(1)_{(a)}) = \Big\{ u(e^{i\theta})\frac{\partial}{\partial\theta} : \;(i)\; u : S^1 \to \mathbb{R} \text{ is continuous, vanishing at } (\pm 1, -i);\; (ii)\; F_u(x) = \tfrac{1}{2}(x^2+1)\,u\Big(\frac{x-i}{x+i}\Big) \text{ is in } \Lambda^*(\mathbb{R}) \Big\}\,. \tag{6.4}$$

We will say that a continuous function $u : S^1 \to \mathbb{R}$ is in *the Zygmund class* $\Lambda^*(S^1)$ *on the circle*, if, after adding the requisite $(ce^{i\theta} + \bar{c}e^{i\theta} + b)$ to normalize u, the function satisfies the conditions in (6.4).

6.2. *Tangent space to the complex-analytic model*

A tangent vector at 0 (the identity mapping) to $T(1)_{(b)}$ corresponds to a one-parameter family F_t of univalent functions (each allowing quasiconformal extension):

$$F_t(z) = z + \frac{b_1(t)}{z} + \frac{b_2(t)}{z^2} + \frac{b_3(t)}{z^3} + \cdots, \quad \text{in } |z| > 1\,, \tag{6.5}$$

with

$$b_k(t) = t\dot{b}_k(0) + o(t), \quad k = 1,2,3,\ldots \tag{6.6}$$

The sequences $\{\dot{b}_k(0),\ k \geq 1\}$ arising this way uniquely correspond to the tangent vectors.

Applying Ahlfors' deep infinitesimal theory for solutions of the Beltrami equation [1], Nag [57] was able to characterize which sequences occur in (6.5). To announce his result, let us expand as a Fourier series the vector field $V[\mu]$:

$$u(e^{i\theta}) = \frac{\dot{w}[\mu](e^{i\theta})}{ie^{i\theta}} = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}. \tag{6.7}$$

Since u is real valued, one knows $a_{-k} = \bar{a}_k, k \geq 1$. The coefficients a_o and $a_{\pm 1}$ do not matter owing to the $sl(2, \mathbb{R})$ normalization.

Nag [57] established the following interesting identities between the coefficients a_k in (6.7) and $\dot{b}_k$ in (6.6):

$$\dot{b}_k(0) = ia_{-k} = i\bar{a}_k, \quad \text{for every} \quad k \geq 2. \tag{6.8}$$

This immediately implies a precise description of the tangent space to $T(1)$ in the complex-analytic model: In (6.5) precisely those sequences $(\dot{b}_1(0), \dot{b}_2(0), \dot{b}_3(0), \dots)$ occur for which the function

$$u(e^{i\theta}) = i\sum_{k=1}^{\infty} \bar{\dot{b}}_k(0)e^{ik\theta} - i\sum_{k=1}^{\infty} \dot{b}_k(0)e^{-ik\theta} \tag{6.9}$$

is in the Zygmund class on S^1.

7. The almost complex structure of $T(1)$

The Lie algebra of the Lie group $\text{Diff}(S^1)$ is the Lie algebra $\text{Vect}(S^1)$ of smooth vector fields on S^1. The complexification $\text{Vect}_{\mathbb{C}}(S^1)$ of $\text{Vect}(S^1)$ is generated by the Fourier modes

$$L_n = e^{in\theta}\frac{d}{d\theta} = iz^{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z} \tag{7.1}$$

with $z = e^{i\theta}$. To $\text{Vect}_{\mathbb{C}}(S^1)$ there does not correspond any global Lie group, yet, Neretin [61] has constructed a complex semigroup whose tangent cone is a convex cone in $\text{Vect}_{\mathbb{C}}(S^1)$. The Lie bracket of $\text{Vect}_{\mathbb{C}}(S^1)$ is given by the Witt law

$$[L_m, L_n] = i(n-m)L_{m+n}. \tag{7.2}$$

A tangent vector to the orbit space $M = \text{Diff}(S^1)/\text{Möb}(S^1)$ at its origin is a linear combination

$$\vartheta = \sum_{m\neq 0,\pm 1} \vartheta_m L_m, \qquad \bar{\vartheta}_m = \vartheta_{-m}, \tag{7.3}$$

where $\vartheta = u(\theta)\,\partial/\partial\theta$ is the corresponding smooth real vector field on the circle and the ϑ_m are the Fourier coefficients of $u(\theta)$. The Lie algebra corresponding to the three missing modes $\vartheta_{-1}, \vartheta_0, \vartheta_1$, is $sl(2;\mathbb{R})$, of course. We may conjugate the series (7.3) by the conjugation operator J to

$$J\vartheta = \sum_{m\neq 0,\pm 1} -i\,\text{sgn}(m)\vartheta_m L_m. \tag{7.4}$$

This is again a smooth vector field, but J can be applied to a much wider class: A classical result of Zygmund [88] says that conjugation of Fourier series preserves the Zygmund class $\Lambda^*(S^1)$.

Notice that $J^2 = -\mathrm{id}$. Kerckhoff (unpublished) first pointed out the fact that the conjugation operation on Zygmund class vector fields on S^1 transmutes to the almost complex structure of $T(1)$. Nag [57] applied the identities (6.8) to give a simple proof of this fact. Indeed, we need to prove that the vector field $V[\mu]$ in (6.2) is related to $V[i\mu]$ as a pair of conjugate Fourier series. But the tangent vector represented by μ in the complex-analytic description of $T(1)$ corresponds to a sequence $(\dot{b}_1(0), \dot{b}_2(0), \dot{b}_3(0), \dots)$, as explained above. Since the b_k are holomorphic in μ, the tangent vector represented by $i\mu$ corresponds to $(i\dot{b}_1(0), i\dot{b}_2(0), i\dot{b}_3(0), \dots)$. The relation (6.8) immediately shows that the kth Fourier coefficient of $V[i\mu]$ is $-i\,\mathrm{sgn}(k)$ times the kth Fourier coefficient of $V[\mu]$, as required.

8. The Bers embedding of $T(1)$

To provide a system of complex coordinates for $T(1)$ Bers [11] embedded $T(1)$ as a holomorphically convex domain into the complex Banach space B which consists of all functions $\phi(z)$, holomorphic in the lower half-plane, $\mathbb{L}$, with bounded *Nehari norm* defined by

$$\|\phi\| = \operatorname*{ess\,sup}_{z\in\mathbb{L}} 4\,|y^2\phi(z)|. \tag{8.1}$$

In the complex-analytic model of $T(1)$, let f^μ represent a point of $T(1)$. We think of f^μ as a quasiconformal map which is univalent in $\mathbb{L}$. The Bers embedding $T(1) \hookrightarrow B$ is defined by

$$f^\mu(z) \mapsto S(f^\mu)(z), \quad z \in \mathbb{L}, \tag{8.2}$$

where S is the Schwarzian derivative as in (1.14) which annihilates Möbius moves according to (1.13).

Since an element f^μ is determined by the Beltrami differential μ up to a Möbius move, we may think of the Bers embedding as a function of μ as well. It then defines a holomorphic embedding of $T(1)$ into B with respect to the complex structure of the Beltrami differentials.

It is an interesting problem to study the locus of $T(1)$ in B in the Bers embedding. In particular, we may compare the locus of $T(1)$ to the bigger locus S in B of the Schwarzian derivatives of *all* univalent maps on $\mathbb{L}$. The following facts are known:

(i) In the Nehari norm, $T(1)$ contains an open ball of radius 2 and is contained in a closed ball of radius 6 [49, 54].

(ii) $T(1)$ is an open set in B while S is closed in B, and the closure of $T(1)$ is a proper subset of S (Gehring [35]).

(iii) The interior of S is $T(1)$ (Gehring [36]).

(iv) $T(1)$ is connected (Earle-Eells [29]) while S contains isolated points (Astala [5], Astala-Gehring [6]).

(v) Suppose that h is a univalent map of $\mathbb{L}$ onto a simply connected domain D of hyperbolic type in $\mathbb{C}$. Then $S(f)$ is in the closure of $T(1)$ if and only if for each

$K > 1$ there exists a homeomorphism g of D onto a quasidisc such that for each disc Q in D, $g|Q$ has a K-quasiconformal extension to $\widehat{\mathbb{C}}$ (Astala and Gehring [7]).

(vi) $T(1)$ is contractible [29] but not star-shaped (Krushkal [48]).

Tukia [79] has shown how to embed $T(1)$ as a real analytic convex domain in a real Banach space.

9. The Kähler structure of $T(1)$

Nag and Verjovsky [59] proved that the natural inclusion

$$M = \mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1) \hookrightarrow T(1) \tag{9.1}$$

is holomorphic. The proof amounts to showing that, if we write in (7.4)

$$J\vartheta = u^*(\theta)\,\partial/\partial\theta, \tag{9.2}$$

then u^* is essentially the Hilbert transform of u; this is not very difficult.

A more subtle result of Nag [59] endows a subspace of $T(1)$ with a Kähler structure and shows the inclusion (9.1) to be a Kähler isometry onto its image. Recall that the existence of a symplectic form ω on M is predicted by the theory of coadjoint orbits. To compute it explicitly, we impose the condition $d\omega = 0$, or, equivalently, at the origin

$$\omega([L_m, L_n], L_p) + \omega([L_n, L_p], L_m) + \omega([L_p, L_m], L_n) = 0. \tag{9.3}$$

Also, ω must vanish whenever one of its arguments is $L_0, L_{\pm 1}$ since these vector fields give the zero tangent vector to M. The conditions (7.2) and (9.3) now lead to a system of difference equations whose only possible solution readily yields a homogeneous Kähler form ω which is given at the origin by

$$\omega(L_m, L_n) = \alpha(m^3 - m)\delta_{m,-n}, \quad m, n \in \mathbb{Z} \setminus \{0, \pm 1\}. \tag{9.4}$$

The constant $\alpha \in \mathbb{C} \setminus \{0\}$ is arbitrary.

Let $v = \sum_m v_m L_m$ and $w = \sum_m w_m L_m$ of the form (7.3) represent two tangent vectors to M at the origin. Then the Kähler metric g, whose Kähler form ω was determined above, assigns the inner product

$$g(v, w) = \mathrm{Re}\Big(\sum_{m=2}^{\infty} v_m \overline{w}_m (m^3 - m)\Big). \tag{9.5}$$

According to standard results in harmonic analysis, the Fourier coefficients of a $C^{k+\epsilon}$ smooth function on S^1 decay at least as fast as $1/n^{k+\epsilon}$. Hence, the infinite series in (9.5) converges absolutely whenever the vector fields v and w are $C^{3/2+\epsilon}$ smooth on S^1 for any $\epsilon > 0$. Zygmund class functions are not necessarily smooth at all, so that the series (9.5) does not yield a well-defined inner product on all of $T(1)$. Claims have been made, though, that even Homeo(S^1)/Möb(S^1) carries a Kähler structure in some sense [67].

The Kähler structure ω is universal in the sense that it is closely related to the Weil–Petersson Kähler forms on each $T(G)$. However, the relationship is not by simple restriction of domains from the infinite-dimensional space $T(1)$ to the complex-analytic subspace $T(G)$, because $T(G)$ is *transversal* to the leaf M of the foliation of $T(1)$ in the following sense: Let us use the geometric definition of $T(1)$ as the space of Möbius-normalized quasidiscs. Bowen [16] proved the deep result that if G uniformizes a compact Riemann surface, then every non-origin point of $T(G)$ corresponds to a quasidisc with *fractal* boundary. On the other hand, the quasidiscs corresponding to points of M are the ones with C^∞ boundaries (Kirillov [45]). Nag [59] showed that every non-null tangent vector to $T(G)$ at the origin produces a vector field on S^1 that cannot be even $C^{3/2+\epsilon}$ smooth.

Nonetheless, the expression of the metric (9.5) is formally the same as that of the Weil–Petersson metric even when it diverges, and it can be regulated as explained by Nag in [59]. This procedure is not entirely satisfactory from the point of the physicists' wish-list, however.

10. Curvature properties of Diff(S^1)/Möb(S^1)

Conformal field theory [44] suggests that the natural value of the constant α is $\alpha = \frac{1}{12}$; see Atiyah [8] for a purely topological derivation. This normalization is natural also when ω is viewed as the generator of the second Gelfand–Fuks cohomology $H^2(\mathrm{Vect}^\infty(S^1);\mathbb{C})$ (Segal [74]). Besides being the unique symplectic form on M, the 2-cocycle ω can also be found as the unique central extension of the Lie algebra of vector fields on the circle. The centrally extended Lie algebra then is called the *Virasoro algebra*. It is customary to write $\alpha = \frac{1}{12}c$ so that the Virasoro law reads as

$$[L_m, L_n] = i(n-m)L_{m+n} + \tfrac{1}{12}c(m^3 - m)\delta_{m,-n}. \tag{10.1}$$

Remarkably, the spectrum of the values of the coupling constant c admitting unitary representations of the Virasoro algebra (10.1) is continuous precisely for $c > 1$ [38]. For $c < 1$, the spectrum consists of the discrete series

$$c = 1 - \frac{6}{(k+2)(k+3)}, \quad k = 1, 2, 3, \ldots \tag{10.2}$$

This phenomenon also shows that the value $c = 1$, i.e., $\alpha = \frac{1}{12}$ is critical.

The Ricci curvature of the Kähler manifold M has been computed by Bowick and Lahiri [18]. The method of Toeplitz operators for dealing with infinite-dimensional Ricci curvature was introduced by Freed [30]. Such computations are surprising because an infinite-dimensional trace can be performed without any regularization. For the critical normalization $\alpha = \frac{1}{12}$ one obtains

$$\mathrm{Ricci} = -26 \times \omega. \tag{10.3}$$

It is surprising to see the mysterious critical dimension $D = 26$ of bosonic string theory emerge in this context! The critical occurrence of the number 26 in two seemingly disparate roles must be an instance of the subtle interplay of Feynman's quantization and geometric quantization of bosonic string theory rather than a mere numerical coincidence, yet this phenomenon has never been geometrically explained.

The Kähler structures of the orbit space $N = \mathrm{Diff}(S^1)/S^1$ form a two-parameter family:

$$\omega(L_m, L_n) = (am^3 + bm)\,\delta_{m,-n}. \tag{10.4}$$

This is non-degenerate when either $a = 0, b \neq 0$, or $a \neq 0, -b/a \neq n^2$ with $n \in \mathbb{Z}$. For $a = 0$, the infinite-dimensional trace in the Ricci curvature of (N, ω) diverges, while for $a \neq 0$, it is finite and the result is

$$R_{\bar{m}n} = (-\tfrac{26}{12}m^3 + \tfrac{1}{6}m)\delta_{m,n}. \tag{10.5}$$

In fact, this computation was made earlier than that for M by several authors [17, 19-22, 64, 86]. Mickelsson [52, 53] extended the formula (10.5) to the case of a string moving on a simple compact Lie group.

The formula (10.5) has been extended to the supersymmetric set-up as well [41, 62, 68, 73, 85]. Then the critical dimension $D = 10$ of superstring theory arises in an equally mysterious manner. The notion of universal super-Teichmüller space seems not to have been developed, though.

11. Holomorphic embedding of $T(1)$ in the universal Siegel disc

Consider the Sobolev space $\mathcal{H} = H^{1/2}(S^1, \mathbb{R})/\mathbb{R}$ of all $H^{1/2}$ real functions on the circle modulo the constant maps and its complexification $\mathcal{H}_{\mathbb{C}} = H^{1/2}(S^1, \mathbb{C})/\mathbb{C}$. Harmonic analysis tells us that the Fourier series

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} u_n e^{in\theta} \tag{11.1}$$

of a $H^{1/2}$ function f converges quasi-everywhere, i.e., off some set of capacity zero. We may think of $\mathcal{H}_{\mathbb{C}}$ equivalently as the space $\ell_2^{1/2}$ of complex sequences $(\ldots, u_{-3}, u_{-2}, u_{-1}, u_0 = 0, u_1, u_2, u_3, \ldots)$ such that $\{\sqrt{|n|}u_n\}$ is square summable. The Hilbert transform on $T(1)$ given by (7.4) also extends to $\mathcal{H}_{\mathbb{C}}$.

The fundamental orthogonal decomposition of $\mathcal{H}_{\mathbb{C}}$ is given by

$$\mathcal{H}_{\mathbb{C}} = W_+ \oplus W_-\,, \tag{11.2}$$

where W_+ (resp. W_-) consists of those functions $f \in \mathcal{H}_{\mathbb{C}}$ whose negative (resp. positive) index Fourier coefficients vanish.

We may provide $\mathcal{H}$ with a symplectic structure, i.e., a non-degenerate skew-symmetric bilinear form S whose formula is

$$S(f,g) = \frac{1}{2\pi} \int\limits_{S^1} f(e^{i\theta}) \frac{d}{d\theta} g(e^{i\theta})\, d\theta. \tag{11.3}$$

The same formula extends to $\mathcal{H}_{\mathbb{C}}$ as well. The W_+, resp. W_-, are the $-i$, resp. $+i$, eigenspaces of the Hilbert transform. Let $f_\pm$ denote the projection of f to $W_\pm$. The inner product on $\mathcal{H}_{\mathbb{C}}$ is given by

$$\langle f, g \rangle = iS(f_+, \bar{g}_+) - iS(f_-, \bar{g}_-). \tag{11.4}$$

Then $\mathrm{QS}(S^1)$ acts faithfully on $\mathcal{H}$. The action V of $\phi \in \mathrm{QS}(S^1)$ on $f \in \mathcal{H}$ is given by

$$V_\phi(f) = f \circ \phi - \frac{1}{2\pi} \int\limits_{S^1} f \circ \phi. \tag{11.5}$$

In fact, the class $\mathrm{QS}(S^1)$ turns out to be the largest possible class of homeomorphisms $\phi : S^1 \to S^1$ for which V_ϕ maps $\mathcal{H}$ to itself. Moreover, Nag and Sullivan [58] show that the action V preserves the canonical symplectic form S. On the other hand, up to a constant multiple, S turns out to be the unique $\mathrm{Möb}(S^1)$-invariant, *a fortiori*, the unique $\mathrm{QS}(S^1)$-invariant symplectic form on $\mathcal{H}$.

Hence, $\mathrm{QS}(S^1)$ becomes a subgroup of the group of real symplectic automorphisms of the symplectic space $(\mathcal{H}, S)$. Moreover, $\mathrm{Sp}(\mathcal{H})/U(1)$ contains $T(1) = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$ as an immersed subspace.

A *polarization* of the space $\mathcal{H}$ with respect to S is a decomposition $\mathcal{H}_{\mathbb{C}} = W \oplus \overline{W}$ such that the complexification of S takes zero values on arbitrary pairs from W. The subspace W is said to be *isotropic* for S. The assignment

$$\langle w_1, w_2 \rangle = -iS(\bar{w}_1, w_2) \tag{11.6}$$

is a Hermitian inner product on W, and the decomposition is a *positive polarization* if (11.6) is positive definite. In this case, W, its conjugate $\overline{W}$, and hence $\mathcal{H}_{\mathbb{C}}$ itself, can be completed to Hilbert spaces with respect to the above Hermitian inner product. We may identify a positive polarization with the isotropic subspace W determining it. The canonical positive polarization is given by W_+.

Note the fundamental fact that the image under the $\mathbb{C}$-linear extension of a symplectic automorphism of a positive isotropic subspace is again such a subspace. Hence, $\mathrm{Sp}(\mathcal{H}, S)$ acts transitively on the space of all positive polarizations, and the stabilizer subgroup at W is evidently identifiable with the unitary group $U(W, \langle\cdot,\cdot\rangle)$. It follows that the homogeneous space Sp/U can be identified with the family $\mathrm{Pol}(\mathcal{H})$ of positive polarizations of $\mathcal{H}$. Either of these spaces can be easily identified with the *universal Siegel disc*, denoted S_∞:

$S_\infty = \{$ All bounded complex linear operators $Z : W_+ \to W_-$ such that :

(1) Z is symmetric w.r.t. S, $S(Zv, w) = S(Zw, v)$; and

(2) $I - Z^*Z$ is positive definite.$\}$

The identification between S_∞ and Pol($\mathcal{H}$) is by associating to $Z \in S_\infty$ the positive isotropic subspace W which is the graph of the operator Z. (Clearly, the origin in D_∞ corresponds to the canonical polarization of W_+.) We have seen that the universal Siegel disc can be described in the following manners:

$$S_\infty = \mathrm{Sp}\,/U = \mathrm{Pol}\,. \tag{11.7}$$

The positive polarizing subspace W can also be taken to be the $-i$-eigenspace of an arbitrary S-compatible almost complex structure J on $\mathcal{H}$. "S-compatible" means that J acts orthogonally with respect to S and that the inner product $\langle\cdot,\cdot\rangle = S(\cdot, J(\cdot))$ is positive definite. Thus, the set of such J's yields yet another description of S_∞. We stress that the symplectic structure S on $\mathcal{H}$ is completely canonical while J is not.

The *Grassmannian* $\mathrm{Gr}(W_+, \mathcal{H}_\mathbb{C})$ consists of all subspaces of $\mathcal{H}_\mathbb{C}$ that are of type W_+. Clearly, S_∞ is embedded in Gr as a complex subspace. The symplectic form S extends to $\mathrm{Gr}(W_+, \mathcal{H}_\mathbb{C})$. Nag and Sullivan [58] (see also Nag [55, 56]) showed that the following chain of mappings consists of equivariant holomorphic symplectomorphisms:

$$M \to T(1) \underset{\Pi}{\to} S_\infty \to \mathrm{Gr}\,. \tag{11.8}$$

The problem of describing the image of Π in S_∞ is an infinite-dimensional analogue of the classical Schottky problem. Nag and Sullivan [58] provided a characterization of the locus of Π as the space of *multiplication-closed* polarizing subspaces W in $\mathcal{H}_\mathbb{C}$. This means that for every $f, g \in W$ such that the pointwise product function fg minus its mean value is in $\mathcal{H}_\mathbb{C}$, that product is actually in the given subspace W. This condition has a natural interpretation in terms of Connes' non-commutative geometry [24, 25, 58] but this would bring us too far. In view of Shiota's theorem [76], one expects a "Novikov conjecture", possibly in some non-commutative sense. The locus of $T(1)$ in S_∞ should be determined within $\mathrm{Gr}(W_+, \mathcal{H}_\mathbb{C})$ as those points $W \in \mathrm{Gr}(W_+, \mathcal{H}_\mathbb{C})$ whose "tau function" [69, 75] satisfies some special conditions related to the Korteweg-de Vries (KdV) hierarchy of equations. That would tie in with the finite-dimensional Novikov conjecture.

Such developments are beyond our scope, yet we briefly indicate how the classical KdV equation for a smooth function $u = u(z, t)$,

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial z^3} + 6u\frac{\partial u}{\partial z}\,, \tag{11.9}$$

arises in the study of M. Indeed, the Lax form of (11.9) is

$$\partial H_u/\partial t = 3[P_u, H_u]\,, \tag{11.10}$$

where H_u is the Hill operator $H_u = (d/dz)^2 + u(z)$ as in (2.5) and

$$P_u = \tfrac{4}{3}\partial_z^3 + u\partial_z + \partial_z u. \tag{11.11}$$

Thus, the KdV equation can be interpreted as describing flows that correspond to isospectral deformations of the Hill operator [9, 74, 75].

12. Conclusion

There seems to be emerging a fascinating interplay of Teichmüller theory and non-commutative geometry which may shed new light on crucial issues of non-perturbative string theory. We hope that our survey will be helpful for someone who is seeking his or her way through the maze of existing literature before tackling the forthcoming papers of Connes and Sullivan [25], Nag and Sullivan [58], and Penner (in preparation).

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